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A filled function method for finding a global minimizer on global integer optimization ☆

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Abstract

The paper gives a definition of the filled function for nonlinear integer programming. This definition is modified from that of the global convexized filled function for continuous global optimization. A filled function with only one parameter which satisfies this definition is presented. We also discuss the properties of the proposed function and give a filled function method to solve the nonlinear integer programming problem. The implementation of the algorithm on several test problems is reported with satisfactory numerical results.

Keywords: Integer programming; Local minimizer; Global minimizer; Filled function; Global optimization

1. Introduction

Global optimization problems arise in many disperse fields of science and technology. The existence of multiple local minima of a general nonconvex objective function makes global optimization a great challenge (see [1,10]). Many approaches using an auxiliary function have been proposed to search for finding a global minimizer of the continuous global optimization problems, including the filled function method (see [2,4,5,7,8,12]), the tunneling method (see [6,9]), etc.

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With regard to the nonlinear integer programming problem, the approaches of continuity are presented in Ge's paper [3] and Zhang's paper [11], and a class of auxiliary functions is proposed in [13]. But these auxiliary functions contain two parameters and must satisfy some conditions so as to ensure that the constructed functions have a minimizer which is better than the current local one. In this paper, we give a definition of the filled function for nonlinear integer programming which is modified from that of the global convexized filled function for continuous global optimization (see [5]). We also present a filled function with only one parameter satisfying the new definition as well as a filled function method for nonlinear integer programming.

This paper is organized as follows. In Section 2, some definitions as well as an algorithm searching for the local minimizer are presented. To follow, a filled function with only one parameter is constructed and its properties are discussed in Section 3. In the next section, we extend the form of this new function. In Section 5, we propose a filled function algorithm and report the results of numerical experiments for testing functions. Finally, conclusions are given in Section 6.

2. Preliminaries

We consider the following nonlinear integer programming problem:

$$(\mathrm{P}_I) \quad \begin{cases} \min & f(x), \\ \text{s.t.} & x \in X_I, \end{cases} \tag{2.1}$$

where $X_I \subset I^n$ is a bounded and closed box set which contains more than one point, and I^n is the set of integer points in R^n. Since X_I is a bounded and closed box set, there exists a constant $K > 0$ such that

$$1 \leqslant K = \max_{x^1, x^2 \in X_I} \|x^1 - x^2\| < \infty,$$

where $\|\cdot\|$ is the usual Euclidean norm.

Notice that the formulation in (P_I) allows the set X_I to be defined by equality constraints as well as inequality constraints. Furthermore, when $f(x)$ is coercive, i.e., $f(x) \to \infty$ as $\|x\| \to \infty$, there exists a box which contains all discrete global minimizers of $f(x)$. Thus, the unconstrained nonlinear integer programming problem

$$(\mathrm{UP}_I) \quad \begin{cases} \min & f(x), \\ \text{s.t.} & x \in I^n, \end{cases} \tag{2.2}$$

can be reduced into an equivalent problem formulation in (P_I). In other words, both unconstrained and constrained nonlinear integer programming problems can be considered in (P_I).

To simplify the discussion in this paper, we recall some definitions involved in nonlinear integer programming.

Definition 2.1 (*see [15]*). The set of all axial directions in I^n is defined by $D = \{\pm e_i : i = 1, 2, \ldots, n\}$, where e_i is the ith unit vector (the n-dimensional vector with the ith component equal to one and other components equal to zero).

Definition 2.2 (*see [15]*). For any $x \in I^n$, the neighborhood of x is defined by $N(x) = \{x, x \pm e_i : i = 1, 2, \ldots, n\}$. Let $N^0(x) = N(x) \setminus \{x\}$.

Definition 2.3 (*see [15]*). An integer point $x_0 \in X_I$ is called a local minimizer of $f(x)$ over X_I if, for all $x \in N(x_0) \cap X_I$, $f(x) \geqslant f(x_0)$ holds; an integer point $x_0 \in X_I$ is called a global minimizer of $f(x)$ over X_I if, for all $x \in X_I$, $f(x) \geqslant f(x_0)$ holds. In addition, x_0 is called a strictly local (global) minimizer of $f(x)$ over X_I if, for all $x \in N^0(x_0) \cap X_I$ $(x \in X_I \backslash \{x_0\})$, $f(x) > f(x_0)$ holds.

It is obvious that the global minimizer of $f(x)$ over X_I is the local minimizer of $f(x)$ over X_I.

The local minimizer of $f(x)$ over X_I is obtained by the following Algorithm 1.

Algorithm 1 (*see [13]*). *Step* 1: Choose any integer $x_0 \in X_I$.

Step 2: If x_0 is a local minimizer of $f(x)$ over X_I, then stop; otherwise search the neighborhood $N(x_0)$ and obtain a point $x \in N(x_0) \cap X_I$ such that $f(x) < f(x_0)$.

Step 3: Let $x_0 := x$, go to Step 2.

3. A filled function and its properties

In this section, we propose a filled function of $f(x)$ at a current local minimizer x_1^* and discuss its properties. Let

$$S_1 = \{x \in X_I \colon f(x) \geqslant f(x_1^*)\} \subset X_I,$$

$$S_2 = \{x \in X_I \colon f(x) < f(x_1^*)\} \subset X_I.$$

Definition 3.1. $P_{x_1^*}(x)$ is called a filled function of $f(x)$ at a local minimizer x_1^* for nonlinear integer programming if $P_{x_1^*}(x)$ has the following properties:

(i) $P_{x_1^*}(x)$ has no local minimizer in the set $S_1 \backslash \{x_0\}$. The prefixed point x_0 is in the set S_1, and is not necessarily a local minimizer of $P_{x_1^*}(x)$.

(ii) If x_1^* is not a global minimizer of $f(x)$, there exists a local minimizer x_1 of $P_{x_1^*}(x)$ such that $f(x_1) < f(x_1^*)$, that is, $x_1 \in S_2$.

Definition 3.1 is different from that of the filled function in [5,13]. It is based on the discrete set in the Euclidean space and x_0 is not necessarily a local minimizer of $P_{x_1^*}(x)$.

Now we propose a one-parameter filled function of $f(x)$ at local minimizer x_1^* as follows:

$$P_{A,x_1^*,x_0}(x) = \eta(\|x - x_0\|) - \varphi(A(1 - \exp(-[\min\{f(x) - f(x_1^*), 0\}]^2))), \tag{3.1}$$

where $A > 0$ is a parameter and the prefixed point x_0 satisfies $f(x_0) \geqslant f(x_1^*)$.

In order to guarantee the theoretical properties of our filled function, $\eta(t)$ and $\varphi(t)$ need to satisfy the following conditions:

(1) $\eta(t)$ and $\varphi(t)$ are strictly monotone increasing functions for any $t \in [0, +\infty)$;

(2) $\eta(0) = 0$, $\varphi(0) = 0$;

(3) $\varphi(t) \to C > 0$ as $x \to +\infty$, where $C \geqslant \max_{x \in X_I} \eta(\|x - x_0\|)$.

In the following we will prove that the above constructed function $P_{A,x_1^*,x_0}(x)$ satisfies the conditions (i) and (ii) of Definition 3.1, i.e., it is a filled function of $f(x)$ at a local minimizer x_1^* satisfying Definition 3.1. First, we give the following Lemma 3.1:

Lemma 3.1. *For any integer point $x \in X_I$, if $x \neq x_0$, there exists $d \in D = \{\pm e_i : i = 1, 2, \ldots, n\}$ such that*

$$\|x + d - x_0\| < \|x - x_0\|. \tag{3.2}$$

Proof. Since $x \neq x_0$, there exists an $i \in \{1, 2, \ldots, n\}$ such that $x_i \neq x_{0i}$. If $x_i > x_{0i}$, then $d = -e_i$; if $x_i < x_{0i}$, then $d = e_i$. □

Theorem 3.1. *$P_{A,x_1^*,x_0}(x)$ has no local minimizer in the set $S_1 \backslash \{x_0\}$ for any $A > 0$.*

Proof. From Lemma 3.1, we know that, for any $x \in S_1$ and $x \neq x_0$, there exists a $d \in D$ such that

$$\|x + d - x_0\| < \|x - x_0\|.$$

Consider the following two cases:

(1) If $f(x_1^*) \leqslant f(x + d) \leqslant f(x)$ or $f(x_1^*) \leqslant f(x) \leqslant f(x + d)$, then

$$\begin{aligned} P_{A,x_1^*,x_0}(x + d) &= \eta(\|x + d - x_0\|) - \varphi(A(1 - \exp(-[\min\{f(x + d) - f(x_1^*), 0\}]^2))) \\ &= \eta(\|x + d - x_0\|) < \eta(\|x - x_0\|) \\ &= \eta(\|x - x_0\|) - \varphi(A(1 - \exp(-[\min\{f(x) - f(x_1^*), 0\}]^2))) \\ &= P_{A,x_1^*,x_0}(x). \end{aligned}$$

Therefore, x is not a local minimizer of function $P_{A,x_1^*,x_0}(x)$.

(2) If $f(x + d) < f(x_1^*) \leqslant f(x)$, then

$$\begin{aligned} P_{A,x_1^*,x_0}(x + d) &= \eta(\|x + d - x_0\|) - \varphi(A(1 - \exp(-[\min\{f(x + d) - f(x_1^*), 0\}]^2))) \\ &= \eta(\|x + d - x_0\|) - \varphi(A(1 - \exp(-[f(x) - f(x_1^*)]^2))) \\ &\leqslant \eta(\|x + d - x_0\|) < \eta(\|x - x_0\|) \\ &= \eta(\|x - x_0\|) - \varphi(A(1 - \exp(-[\min\{f(x) - f(x_1^*), 0\}]^2))) \\ &= P_{A,x_1^*,x_0}(x). \end{aligned}$$

Therefore, x is not a local minimizer of function $P_{A,x_1^*,x_0}(x)$. □

From Theorem 3.1, we conclude that the constructed function $P_{A,x_1^*,x_0}(x)$ satisfies the first property of Definition 3.1 without any further assumption on the parameter A.

Since $X_I = S_1 \cup S_2$, all local minimizers of function $P_{A,x_1^*,x_0}(x)$ except x_0 belong to the set S_2.

It is obvious that if $A = 0$, $P_{A,x_1^*,x_0}(x) = \eta(\|x - x_0\|)$ has a unique local minimizer x_0 in the set X_I. Since $f(x_0) \geqslant f(x_1^*)$, that is, $x_0 \in S_1$, $P_{A,x_1^*,x_0}(x)$ has no local minimizers in the set S_2. In this case, $P_{A,x_1^*,x_0}(x)$ is not a filled function of $f(x)$ at a local minimizer x_1^*. Hence a question arises as to how large the parameter A should be such that $P_{A,x_1^*,x_0}(x)$ has a local minimizer in the set S_2. To answer this question, we have the following theorem.

Theorem 3.2. *Let the set S_2 be nonempty. If the parameter $A > 0$ satisfies the condition*

$$A > \frac{\varphi^{-1}(C)\,\exp([f(x^*) - f(x_1^*)]^2)}{\exp([f(x^*) - f(x_1^*)]^2) - 1}, \tag{3.3}$$

where $C \geqslant \max_{x \in X_I} \eta(\|x - x_0\|)$, x^* *is a global minimizer of* $f(x)$, *then* $P_{A,x_1^*,x_0}(x)$ *has a local minimizer in the set* S_2.

Proof. Since the set S_2 is nonempty and x^* is a global minimizer of $f(x)$, $f(x^*) < f(x_1^*)$ holds and

$$\begin{aligned} P_{A,x_1^*,x_0}(x^*) &= \eta(\|x^* - x_0\|) - \varphi(A(1 - \exp(-[\min\{f(x^*) - f(x_1^*), 0\}]^2))) \\ &= \eta(\|x^* - x_0\|) - \varphi(A(1 - \exp(-[f(x^*) - f(x_1^*)]^2))) \\ &\leqslant C - \varphi(A(1 - \exp(-[f(x^*) - f(x_1^*)]^2))). \end{aligned}$$

Since $\varphi(t)$ is a strictly monotone increasing function for any $t \in [0, +\infty)$, $\varphi^{-1}(t)$ exists. When A satisfies the condition (3.3), we have $P_{A,x_1^*,x_0}(x^*) < 0$.

On the other hand, for any $y \in S_1$, we have

$$\begin{aligned} P_{A,x_1^*,x_0}(y) &= \eta(\|y - x_0\|) - \varphi(A(1 - \exp(-[\min\{f(y) - f(x_1^*), 0\}]^2))) \\ &= \eta(\|y - x_0\|) \geqslant 0. \end{aligned}$$

Therefore the global minimum points of $P_{A,x_1^*,x_0}(x)$ belong to the set S_2. This result, combined with Theorem 3.1, implies the thesis. □

In summary, from Theorems 3.1 and 3.2, if parameter A is large enough, the constructed function $P_{A,x_1^*,x_0}(x)$ does satisfy all the conditions of Definition 3.1, i.e., function $P_{A,x_1^*,x_0}(x)$ is a filled function.

We know the value of $f(x_1^*)$; however, we generally do not know the global minimal value or global minimizer of $f(x)$. So it is difficult to find the lower bound of parameter A presented in Theorem 3.2.

But for practical consideration, problem (P_I) might be solved if we can find an $x \in X_I$ such that $f(x) < f(x^*) + \varepsilon$, where $f(x^*)$ is the global minimal value of problem (P_I), and ε is a given desired optimality tolerance. So we consider the case that the current local minimizer x_1^* satisfies that $f(x_1^*) \geqslant f(x^*) + \varepsilon$. In the following Theorem 3.3 we develop a lower bound of parameter A which depends only on the given optimality tolerance ε.

Theorem 3.3. *Suppose that* ε *is a small positive constant and* $A > 0$ *satisfies the condition*

$$A > \frac{\varphi^{-1}(C)\ \exp(\varepsilon^2)}{\exp(\varepsilon^2) - 1}. \tag{3.4}$$

Then, given any x_1^* *of* $f(x)$ *such that* $f(x_1^*) \geqslant f(x^*) + \varepsilon$, $P_{A,x_1^*,x_0}(x)$ *has at least one local minimizer in the set* S_2, *where* x^* *is a global minimizer of* $f(x)$.

Proof. Since $\exp(t)/(\exp(t) - 1)$ is a strictly monotone decreasing function for any $t \in (0, +\infty)$, and $f(x_1^*) - f(x^*) \geqslant \varepsilon$, we get

$$\frac{\exp([f(x^*) - f(x_1^*)]^2)}{\exp([f(x^*) - f(x_1^*)]^2) - 1} \leqslant \frac{\exp(\varepsilon^2)}{\exp(\varepsilon^2) - 1},$$

that is,

$$\frac{\varphi^{-1}(C)\ \exp([f(x^*)-f(x_1^*)]^2)}{\exp([f(x^*)-f(x_1^*)]^2)-1} \leqslant \frac{\varphi^{-1}(C)\ \exp(\varepsilon^2)}{\exp(\varepsilon^2)-1}. \tag{3.5}$$

It follows from (3.3)–(3.5) and Theorem 3.2 that the conclusion of this theorem holds. □

About a prefixed point $x_0 \in S_1$, we have the following property:

Theorem 3.4. *The prefixed point x_0 is a local minimizer of $P_{A,x_1^*,x_0}(x)$ provided that x_0 is a local minimizer of $f(x)$ or for any $d \in D$, $f(x_0+d) \geqslant f(x_1^*)$.*

Proof. If x_0 is a local minimizer of $f(x)$, then for any $d \in D$, we have

$$f(x_0+d) \geqslant f(x_0) \geqslant f(x_1^*). \tag{3.6}$$

Therefore, if x_0 is a local minimizer of $f(x)$ or for any $d \in D$, $f(x_0+d) \geqslant f(x_1^*)$, we have

$$f(x_0+d) \geqslant f(x_1^*), \quad \forall d \in D. \tag{3.7}$$

It follows from (3.7) that

$$P_{A,x_1^*,x_0}(x_0+d) = \eta(\|d\|) \geqslant \eta(\|x_0-x_0\|) = P_{A,x_1^*,x_0}(x_0).$$

Therefore, the conclusion in this theorem holds. □

We construct the following auxiliary nonlinear integer programming problem (AP_I) related to the problem (P_I):

$$(\text{AP}_I) \quad \begin{cases} \min & P_{A,x_1^*,x_0}(x), \\ \text{s.t.} & x \in X_I. \end{cases} \tag{3.8}$$

It follows from the discussions of Theorems 3.1–3.3 that, if A satisfies (3.4), $P_{A,x_1^*,x_0}(x)$ is a filled function of $f(x)$ at local minimizer x_1^* which satisfies $f(x_1^*) \geqslant f(x^*) + \varepsilon$. Thus, if we use a local minimization method to solve problem (AP_I) from any initial point on X_I, it is obvious that the minimization sequence converges either to the prefixed point x_0 or to a point $x' \in X_I$ such that $f(x') < f(x_1^*)$. If we find such an x', then, using a local minimization method to minimize $f(x)$ on X_I from initial point x', we can find a minimum point $x_2^* \in X_I$ such that $f(x_2^*) \leqslant f(x') < f(x_1^*)$. With x_2^* replacing x_1^*, one can construct a new filled function and then find a much lower minimizer of $f(x)$ in the same way. Repeating the above process, one can finally find the global minimizer x^* of $f(x)$.

4. The extension of the filled function $P_{A,x_1^*,x_0}(x)$

For the box constrained continuous global optimization problem

$$(\text{P}) \quad \begin{cases} \min & f(x), \\ \text{s.t.} & x \in X, \end{cases} \tag{4.1}$$

where $f(x)$ is a continuously differentiable function on X, and X is a bounded and closed box in R^n, [14] presented a class of globally convexized filled functions of $f(x)$ at local minimizer x_1^*. The filled function is described in the following form:

$$U(x) = u(x) - w(Av(x)), \tag{4.2}$$

where $A > 0$ is a parameter, and $u(x)$, $v(x)$ and $w(x)$ satisfy the following three assumptions. Let $X \subset R^n$ be a box set.

Assumption 1 (*see [14]*). $u(x) \geqslant 0$ is a continuously differentiable function on X which has only one minimizer x_0 and for any $x \in X$, $x_0 - x$ is a descent direction of $u(x)$ at x.

Assumption 2 (*see [14]*). $v(x) \geqslant 0$ is a continuously differentiable function on X such that

$$\begin{array}{ll} v(x) = 0 \quad \text{and } \nabla v(x) = 0 & \text{for all } x \in S_1, \\ v(x) > 0 \quad \text{for all } x \in S_2. & \end{array} \tag{4.3}$$

Assumption 3 (*see [14]*). $w(t)$ is a continuously differentiable univariate function defined on $[0, +\infty)$, $w(0) = 0$, and there exists $t_0 > 0$ such that $w(t_0) > C \geqslant \max_{x \in X} u(x)$.

Under the above assumptions, it is difficult to prove that $U(x)$ is a filled function of $f(x)$ at a local minimizer x_1^* for the nonlinear integer programming problem (P_I) which satisfies Definition 3.1 in this paper.

In some special cases, however, we can prove that $U(x)$ is such a filled function satisfying Definition 3.1. For example, let $u(x) = \eta(\|x - x_0\|)$ or $u(x) = \eta(\|x - x_0\|^2)$; we have the following forms of $U(x)$:

$$U^1_{x_1^*, x_0}(x) = \eta(\|x - x_0\|) - w(Av(x)), \tag{4.4}$$

$$U^2_{x_1^*, x_0}(x) = \eta(\|x - x_0\|^2) - w(Av(x)), \tag{4.5}$$

where $\eta(t)$ and $w(x)$ satisfy conditions (1)–(3) mentioned in Section 3, respectively.

The following Theorem 4.1 has a similar proof to that of Theorem 3.1 and Theorems 4.2 and 4.3 to Theorem 3.2. Hence their proofs are omitted.

Theorem 4.1. *For the nonlinear integer programming problem* (P_I), $U^1_{x_1^*, x_0}(x)$ *and* $U^2_{x_1^*, x_0}(x)$ *have no local minimizer in the set* $S_1 \backslash \{x_0\}$ *for any* $A > 0$.

Theorem 4.2. *Let the set* S_2 *be nonempty. For the nonlinear integer programming problem* (P_I), *if the parameter* $A > 0$ *satisfies the condition*

$$A > \frac{w^{-1}(C_1)}{v(x^*)}, \tag{4.6}$$

where $C_1 \geqslant \max_{x \in X_I} \eta(\|x - x_0\|)$, x^* *is a global minimizer of* $f(x)$, *then* $U^1_{x_1^*, x_0}(x)$ *has a local minimizer in the set* S_2.

Theorem 4.3. *Let the set S_2 be nonempty. For the nonlinear integer programming problem* (P_I), *if the parameter $A > 0$ satisfies the condition*

$$A > \frac{w^{-1}(C_2)}{v(x^*)}, \tag{4.7}$$

where $C_2 \geqslant \max_{x \in X_I} \eta(\|x - x_0\|^2)$, x^* is a global minimizer of $f(x)$, then $U^2_{x_1^*, x_0}(x)$ has a local minimizer in the set S_2.

5. Algorithm and numerical results

Based on the theoretical results in the previous sections, a filled function algorithm over X_I similar to that in [13] is proposed as follows.

Algorithm 2 (The filled function method)

Step 1: Given a constant $N_\mathrm{L} > 0$ as the tolerance parameter for terminating the minimization process of problem (P_I) and a small constant $\varepsilon > 0$ as a desired optimality tolerance, choose any x_0 in the set X_I.

Step 2: Starting from x_0, obtain a local minimizer x_1^* of $f(x)$ by implementing Algorithm 1.

Step 3: Construct the filled function $P_{A, x_1^*, x_0}(x)$ as follows:

$$P_{A, x_1^*, x_0}(x) = \eta(\|x - x_0\|) - \varphi(A(1 - \exp(-[\min\{f(x) - f(x_1^*), 0\}]^2))),$$

where A satisfies (3.3) or (3.4). Let $N = 0$.

Step 4: If $N \geqslant N_\mathrm{L}$, then go to Step 7.

Step 5: Set $N = N + 1$. Draw an initial point on the boundary of the set X_I. Starting from this point, minimize $P_{A, x_1^*, x_0}(x)$ on the set X_I using any local minimization method. Suppose that x' is an obtained local minimizer, if $x' = x_0$, go to Step 4; otherwise, go to Step 6.

Step 6: Minimize $f(x)$ on the set X_I from the initial point x', and obtain a local minimizer x_2^* of $f(x)$. Let $x_1^* = x_2^*$ and go to Step 3.

Step 7: Stop the algorithm; output x_1^* and $f(x_1^*)$ as an approximate global minimal solution and global minimal value of problem (P_I), respectively.

Numerical results: Although the focus of this paper is more theoretical than computational, we still test our algorithm on several global minimization problems to have an initial feeling of the practicability of the filled function method.

In Section 4, we proposed two extensive forms of the filled function. However, we have found by numerical testing that the filled function (3.1) is superior to others in computational effectiveness. Accordingly, this filled function is adopted in the following testing example.

Example.

$$\begin{aligned} \min \quad & f(x) = (x_1 - 1)^2 + (x_n - 1)^2 + n \sum_{i=1}^{n-1} (n - i)(x_i^2 - x_{i+1})^2, \\ \text{s.t.} \quad & |x_i| \leqslant 5, \quad x_i \text{ integer}, \quad i = 1, 2, \ldots, n. \end{aligned}$$

Table 1
Results of numerical example: $n=2$, $\varepsilon=0.05$, $A=C\ \exp(\varepsilon^2)/(\exp(\varepsilon^2)-1)\doteq 6065$, $N_L=10^2+1$

T_S	k	x^k_{ini}	$x^k_{\text{f-lo}}$	$f(x^k_{\text{f-lo}})$	$x^k_{\text{p-lo}}$	$f(x^k_{\text{p-lo}})$	QIN
1	1	(−5,−3)	(0,0)	2	(1,1)	0	0
	2	(1,1)	(1,1)	0			$\geqslant 10^2+1$
2	1	(5,5)	(2,3)	7	(1,1)	0	2
	2	(1,1)	(1,1)	0			$\geqslant 10^2+1$
3	1	(−4,3)	(−2,3)	15	(1,1)	0	1
	2	(1,1)	(1,1)	0			$\geqslant 10^2+1$
4	1	(2,3)	(2,3)	7	(1,1)	0	1
	2	(1,1)	(1,1)	0			$\geqslant 10^2+1$

This problem is a box constrained/unconstrained nonlinear integer programming problem. It has 11^n feasible points and many local minimizers (4, 6, 7, 10 and 12 local minimizers for n=2, 3, 4, 5 and 6, respectively) but only one global minimum solution: $x^*_{\text{global}}=(1,1,\ldots,1)$ with $f(x^*_{\text{global}})=0$, for all n. We consider three cases of the problem: n=2, 3 and 5. There are about 1.21×10^2, 1.331×10^3, 1.71×10^5 feasible points for n=2, 3, 5, respectively.

In the following, the proposed solution algorithm is programmed in MATLAB 6.5.1 Release 11 for working on the WINDOWS XP system with 900 MHz CPU. The MATLAB 6.5.1 subroutine is used as the local neighborhood search scheme to obtain local minimizers of $f(x)$ in Step 2 and local minimizers of $P_{A,x_1^*,x_0}(x)$ in Step 5. We choose $\eta(t)=t$, $\varphi(t)=t$; thus, the function $P_{A,x_1^*,x_0}(x)$ is as follows:

$$P_{A,x_1^*,x_0}(x)=\|x-x_0\|-A(1-\exp(-[\min\{f(x)-f(x_1^*),0\}]^2)),$$

where $A=C\ \exp(\varepsilon^2)/(\exp(\varepsilon^2)-1)$, $C=10\sqrt{n}+1$ and let $\varepsilon=0.05$. The tolerance parameter $N_L=10^n$, and n is the variable number of $f(x)$.

The iterative results of the computation for the example are summarized in Tables 1–3 for $n=2,3,5$, respectively. The symbols used are shown as follows:

n	number of variables
T_S	times of tests
k	times for the local minimization process of the problem (P_I)
x^k_{ini}	initial point for the kth local minimization process of problem (P_I)
$x^k_{\text{f-lo}}$	minimizer for the kth local minimization process of problem (P_I)
$f(x^k_{\text{f-lo}})$	value of $x^k_{\text{f-lo}}$
$x^k_{\text{p-lo}}$	minimizer for the kth local minimization process of problem (AP_I)
$f(x^k_{\text{p-lo}})$	value of $x^k_{\text{p-lo}}$
QIN	iteration number for the kth local minimization process of problem (AP_I).

From Table 1, we see that there are many local minimizers of this problem. We take the initial point $x^1_{\text{ini}}=(5,5)$, and by Algorithm 1, we obtain the first local minimizer of this problem $x^1_{\text{f-lo}}=(2,3)$ with $f(x^1_{\text{f-lo}})=7$. Then we construct a filled function $P_{A,x^1_{\text{f-lo}},x^1_{\text{ini}}}(x)$ which has a local minimizer $x^1_{\text{p-lo}}=(1,1)$ in the set $S_2^1=\{x\in X_I : f(x)<f(x^1_{\text{f-lo}})\}$. $x^1_{\text{p-lo}}=(1,1)$ is obtained by Algorithm 1 starting from any

Table 2
Results of numerical example: $n = 3$, $\varepsilon = 0.05$, $A = C\ \exp(\varepsilon^2)/(\exp(\varepsilon^2) - 1) \doteq 7337$, $N_L = 10^3 + 1$

T_S	k	x^k_{ini}	$x^k_{\text{f-lo}}$	$f(x^k_{\text{f-lo}})$	$x^k_{\text{p-lo}}$	$f(x^k_{\text{p-lo}})$	QIN
1	1	(−4,0,4)	(−1,2,3)	17	(−1,1,1)	4	2
	2	(−1,1,1)	(0,0,0)	2	(1,1,1)	0	0
	3	(1,1,1)	(1,1,1)	0			$\geqslant 10^3$+1
2	1	(3,3,3)	(1,2,3)	13	(1,1,1)	0	0
	2	(1,1,1)	(1,1,1)	0			$\geqslant 10^3$+1
3	1	(0,4,4)	(1,2,3)	13	(1,1,1)	0	1
	2	(1,1,1)	(1,1,1)	0			$\geqslant 10^3$+1

Table 3
Results of numerical example: $n = 5$, $\varepsilon = 0.05$, $A = C\ \exp(\varepsilon^2)/(\exp(\varepsilon^2) - 1) \doteq 9356$, $N_L = 10^5 + 1$

T_S	k	x^k_{ini}	$x^k_{\text{f-lo}}$	$f(x^k_{\text{f-lo}})$	$x^k_{\text{p-lo}}$	$f(x^k_{\text{p-lo}})$	QIN
1	1	(0,0,2,0,2)	(0,0,0,0,0)	2	(1,1,1,1,1)	0	0
	2	(1,1,1,1,1)	(1,1,1,1,1)	0			$\geqslant 10^5$+1
2	1	(−2,2,0,1,1)	(−1,1,1,1,1)	4	(0,0,0,0,0)	2	1
	2	(0,0,0,0,0))	(0,0,0,0,0)	2	(1,1,1,1,1)	0	3
	3	(1,1,1,1,1)	(1,1,1,1,1)	0			$\geqslant 10^5$+1
3	1	(0,3,0,3,3)	(1,1,1,2,3)	19	(1,1,1,1,1)	0	8
	2	(1,1,1,1,1)	(1,1,1,1,1)	0			$\geqslant 10^5$+1

initial point in the set X_I after two failed attempts of local search. From this local minimizer, we obtain another local minimizer $x^2_{\text{f-lo}} = (1, 1)$ of problem (P_I) with $f(x^2_{\text{f-lo}}) = 0$. Then, we construct another filled function $P_{A,x^2_{\text{f-lo}},x^2_{\text{ini}}}(x)$. There is no local minimizer of $P_{A,x^2_{\text{f-lo}},x^2_{\text{ini}}}(x)$ obtained after 10^2+1 failed attempts of local search. So we stop Algorithm 2. The global minimizer of problem (P_I) is $x^*_{\text{global}} = (1, 1)$ with $f(x^*_{\text{global}}) = 0$. It can also be obtained if we choose another initial point, such as $x^1_{\text{ini}} = (-4, 3)$, etc.

6. Conclusions

This paper gives a definition of the filled function for the nonlinear integer programming problem, and presents a new filled function which has only one parameter. A filled function algorithm based on this given filled function is designed. The implementation of the algorithm on several test problems is reported with satisfactory numerical results.

References

[1] P.M. Pardalos, H.E. Romeijn, H. Tuy, Recent development and trends in global optimization, J. Comput. Appl. Math. 124 (2000) 209–228.

[2] R.P. Ge, A filled function method for finding a global minimizer of a function of several variables, Math. Programming 46 (1990) 191–204.
[3] R.P. Ge, H. Huang, A continuous approach to nonlinear integer programming, Appl. Math. Comput. 34 (1989) 39–60.
[4] R.P. Ge, Y.F. Qin, A class of filled functions for finding a global minimizer of a function of several variables, J. Optim. Theory Appl. 54 (2) (1987) 241–252.
[5] R.P. Ge, Y.F. Qin, The global convexized filled functions for globally optimization, Appl. Math. Comput. 54 (2) (1990) 131–158.
[6] A.V. Levy, A. Montalvo, The tunneling algorithm for the global minimization of function, SIAM J. Sci. Statist. Comput. 6 (1) (1985) 15–29.
[7] X. Liu, A class of generalized filled functions with improved computability, J. Comput. Appl. Math. 137 (1) (2001) 62–69.
[8] S. Lucid, V. Piccialli, New classes of globally convexized filled functions for global optimization, J. Global Optim. 24 (2002) 219–236.
[9] E.M. Oblow, SPT: a stochastic tunneling algorithm for global optimization, J. Global Optim. 20 (2001) 195–212.
[10] L.C.W. Dixon, J. Gomulka, S.E. Herson, Reflection on global optimization problems, in: L.C.W. Dixon (Ed.), Optimization in Action, Academic Press, New York, 1976, pp. 398–435.
[11] L.S. Zhang, F. Gao, Y.R. Yao, Continuity methods for nonlinear integer programming, OR Trans. 2 (2) (1998) 59–66.
[12] L.S. Zhang, C. Ng, D. Li, W.W. Tian, A new filled function method for global optimization, J. Global Optim. 28 (2004) 17–43.
[13] W.X. Zhu, A filled function method for nonlinear integer programming, Chinese ACTA Math. Appl. Sinica 23 (4) (2000) 481–487.
[14] W.X. Zhu, On the globally convexized filled function method for box constrained continuous global optimization, appeared in Optimization On Line March 2003 http://www.optimization-online.org/DB-FILE/2004/03/846.pdf.
[15] W.X. Zhu, L.S. Zhang, An approximate algorithm for nonlinear integer programming, OR Trans. 1 (1) (1997) 72–81.